Decay of Dirac Massive Hair in the Background of Spherical Black Hole

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The intermediate and late-time behaviour of massive Dirac hair in the static spherically general black hole spacetime was studied. It was revealed that the intermediate asymptotic pattern of decay of massive Dirac spinor hair is dependent on the mass of the field under consideration as well as the multiple number of the wave mode. The long-lived oscillatory tail observed at timelike infinity in the considered background decays slowly as $t^{-5/6}$.

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I. INTRODUCTION

Late-time behaviour of various fields in the spacetime of a collapsing body plays an important role in black hole physics. It happens that regardless of details of the collapse or the structure and properties of the collapsing body the resultant black hole can be described only by few parameters such as mass, charge and angular momentum, *black holes have no hair*. On its own, it is interesting to investigate how these hair loss proceeds dynamically.

The research in this direction have quite long history. Price in [1] showed that the late-time behavior is dominated by the factor $t^{-(2l+3)}$, for each multipole moment l. It was revealed in Ref. [2] that the decay-rate along null infinity and along the future event horizon was governed by the power laws $u^{-(l+2)}$ and $v^{-(l+3)}$, where u and v were the outgoing Eddington-Finkelstein (ED) and ingoing ED coordinates. On the other hand, Ref. [3] was devoted to the scalar perturbations on Reissner-Nordström (RN) background for the case when $\mid Q \mid < M$ has the following dependence on time $t^{-(2l+2)}$, while for $\mid Q \mid = M$ the late-time behavior at fixed r is governed by $t^{-(l+2)}$.

A charged hair turned out to decay slower than a neutral one [4]-[6]. The late-time tails in gravitational collapse of a self-interacting (SI) fields in the background of Schwarzschild solution was reported by Burko [7] and in RN solution at intermediate late-time was considered in Ref. [8]. At intermediate late-time for small mass m the decay was dominated by the oscillatory inverse power tails $t^{-(l+3/2)}\sin(mt)$. This analytic prediction was verified at intermediate times, where $mM \ll mt \ll 1/(mM)^2$. It was proved analytically [9] that for a nearly extreme RN spacetime the inverse power law behavior of the dominant asymptotic tail is of the form $t^{-5/6}\sin(mt)$. The asymptotic tail behaviour of SI scalar field was also studied in Schwarzschild spacetime [10]. The oscillatory tail of scalar field has the decay rate of $t^{-5/6}$ at asymptotically late time. The power-law tails in the evolution of a charged massless scalar field around a fixed background of dilaton black hole was studied in Ref. [11], while the case of a self-interacting scalar field was elaborated in [12]. The analytical proof of the above mentioned behaviour of massive scalar hair in the background of dilaton black hole in the theory with arbitrary coupling constant between $U(1)$ gauge field and dilaton field was given in Ref. [22].

On its own, the late-time behaviour of massive Dirac fields were studied in the spacetime of Schwarzschild black hole [13]. It was found that the asymptotic behaviour of these fields is dominated by a decaying tail without any oscillation. The dumping exponent was independent on the multiple number of the wave mode and on the mass of the Dirac field. It also happened that the decay of the massive Dirac fields was slower comparing the decay of massive scalar field. The above analysis was supplemented by the studies of charged massive Dirac fields in the spacetime of RN black hole [14]. The case of the decay of the fields in the stationary axisymmetric black hole background was studied numerically in Ref. [15] and it was found that in the case of Kerr black hole that the oscillatory inverse-power law of the dominant asymptotic tail behaviour is approximately depicted by the relation $t^{-5/6}\sin(mt)$. In Ref. [16] both the intermediate late-time tail and the asymptotic behaviour of the charged massive Dirac fields in the background of Kerr-Newmann black hole was investigated. It was demonstrated that the intermediate late-time behaviour of the fields under consideration is dominated by an inverse power-law decaying tail without any oscillation.

The late-time behaviour of massive vector field obeying the Proca equation of motion in the background of Schwarzschild black hole was studied in [17]. It was revealed that at intermediate late times, three functions characterizing the field have different decay law depending on the multiple number l. On the contrary, the late-time behaviour is independent on l, i.e., the late-time decay law is proportional to $t^{-5/6} \sin(mt)$. In Ref. [18] the analytical studies concerning the intermediate and late-time decay pattern of massive Dirac hair on the dilaton black hole were conducted. Dilaton black hole constitutes a static spherically symmetric solution of the theory being the low-energy limit of the string theory with arbitrary coupling constant α.

For the case of n-dimensional static black holes, the *no-hair* theorem is quite well established [19]. The mechanism of decaying black hole hair in higher dimensional static black hole case concerning the evolution of massless scalar field in the n-dimensional Schwarzschild spacetime was determined in Ref. [20]. The late-time tails of massive scalar fields in the spacetime of n-dimensional static charged black hole was elaborated in Ref. [21] and it was found that the intermediate asymptotic behaviour of massive scalar field had the form $t^{-(l+n/2-1/2)}$. This pattern of decay was checked numerically for $n = 5, 6$. Quasi-normal modes for massless Dirac field in the background of n-dimensional Schwarzschild black hole were obtained in [23], while in Ref. [24] studies of massless fermion excitations on a tensional three-brane embedded in six-dimensional spacetime were provided. On the other hand, the intermediate and late-time decay of massive scalar hair on static brane black holes was elaborated in [25], while the decay of massive Dirac hair in the spacetime of black hole in question was studied in [26].

At the beginning of our universe several phase transitions leading to the topological defect formations might happened [27]. The interactions of topological defects such as global monopoles or cosmic strings with compact objects like black holes attract much attention and their physical characteristics are widely studied in literature. For instance, the black hole global monopole system has an unusual topological property of possessing a solid deficit angle [28]. The decay of massive scalar hair in the background of a Schwarzschild black hole with global monopole was considered in Ref. [29]. It happened that the topological defects makes the massive scalar field hair decay faster in the intermediate regime comparing to the decay of such hair on the black holes without defects. On the other hand, the late-time behaviour was unaffected by the presence of global monopole. The Schwarzschild black hole global monopole system coupled to scalar fields was elaborated in Ref. [30].

Cosmic strings also acquire much interests as well as the cosmic string black hole systems. The metric of such a system assuming the distributional mass source was derived in [31] (the so-called *thin string limit*). Then it was revealed that it constituted the limit of much more realistic situation when black hole was pierced by a Nielsen-Olesen vortex [32]. Especially interesting behaviour was found in the case of extremal black hole. Namely, for some range of black hole parameters it was shown that extremal black hole expelled the vortex (the so called *Meissner effect*). It turned our that in the case of extremal black holes in dilaton gravity one has always expulsion of the Higgs fields from their interiors [33]. The intermediate and late-time decay patterns of massive Dirac hair in the spacetime of black hole and topological defects were elaborated in Ref. [34]. The system of black hole with a global monopole and dilaton black hole pierced by a cosmic string was considered. Among all it was revealed that the late-time asymptotic decay of massive Dirac hair in the systems in question were proportional to $t^{-5/6}$ and topological defects did not affect this relation.

In our paper we shall discuss the intermediate and late-time behaviour of the massive Dirac spinor field in the spacetime of static spherically symmetric general kind of black hole. We revealed that the intermediate asymptotic behaviour is not the final pattern of the decay of the massive spinor hair. In Sec. II we present and summary our analytic arguments concerning the decay of massive spinor Dirac hair in the background of the black hole under consideration. In Sec. III we conclude that the long-lived oscillatory tail is generally observed at timelike infinity in static spherically symmetric black hole spacetime which decay pattern is proportional to $t^{-\frac{5}{6}}$.

II. MASSIVE DIRAC HAIR ON SPHERICALLY SYMMETRIC BLACK HOLE

A. Properties of the Dirac Equation

The mass Dirac equation in a background of spherically symmetric black hole is given by

$$\left(\gamma^\mu \nabla_\mu \psi - m \right) \psi = 0, \tag{1}$$

where ∇_μ is the covariant derivative $\nabla_\mu = \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_a \gamma_b$, μ and a are tangent and spacetime indices. There are related by e_μ^a, a basis of orthonormal one-forms. The quantity $\omega_\mu^{ab} \equiv \omega^{ab}$ are the associated connection one-forms satisfying $de^a + \omega_b{}^a \wedge e^b = 0$, while the gamma matrices fulfill the relation $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. Now we shall recall some basic properties of the Dirac operator $\not{D} = \gamma^\mu \nabla_\mu$ on an n-dimensional manifold (see e.g., [18]). In what follows we

assume that the metric of the underlying spacetime may be rewritten as a product of the form

$$g_{\mu\nu}dx^\mu dx^\nu = g_{ab}(x)dx^a dx^b + g_{mn}(y)dy^m dy^n. \tag{2}$$

The above metric decomposition will be subject to the direct sum of the Dirac operator, namely one obtains

$$\slashed{D} = \slashed{D}_x + \slashed{D}_y. \tag{3}$$

If we define a Weyl conformal rescaling as

$$g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}, \tag{4}$$

it consequently provides that one gets

$$\slashed{D}\psi = \Omega^{-\frac{1}{2}(n+1)}\tilde{\slashed{D}}\tilde{\psi}, \qquad \psi = \Omega^{-\frac{1}{2}(n-1)}\tilde{\psi}. \tag{5}$$

Having in mind that spherically symmetric form of the line element provides also conformal flatness for a static metric, one obtains

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 d\Sigma_{n-2}^2, \tag{6}$$

where $A = A(r)$, $B = B(r)$, $C = C(r)$ are functions only of the radial variable r, and the *transverse* metric $d\Sigma_{n-2}^2$ is independent on t and on r.

Suppose then, that Ψ is a spinor eigenfunction on the $(n-2)$-dimensional *transverse* manifold Σ. It leads to the following:

$$\slashed{D}_\Sigma \Psi = \lambda \Psi. \tag{7}$$

In case of $(n-2)$-dimensional sphere the eigenvalues for spinor Ψ, where found in Ref.[35]. They imply

$$\lambda^2 = \left(l + \frac{n-2}{2}\right)^2, \tag{8}$$

where $l = 0, 1, \ldots$ Using the properties given above one may also assume that the following is satisfied:

$$\slashed{D}\psi = m\psi. \tag{9}$$

It enables us to set the form of the spinor ψ, i.e.,

$$\psi = \frac{1}{A^{\frac{1}{2}}} \frac{1}{C^{\frac{(n-2)}{2}}} \chi \otimes \Psi. \tag{10}$$

Next, the explicit calculations reveal that we arrive at the relation

$$(\gamma^0 \partial_t + \gamma^1 \partial_y)\chi = A(m - \frac{\lambda}{C})\chi, \tag{11}$$

where we have introduced the *radial optical distance* (i.e., the Regge-Wheeler radial coordinate) $dy = B/Adr$ and γ^0, γ^1 satisfy the Clifford algebra in two spacetime dimensions.

We remark that an identical result may be obtained if a Yang-Mills gauge field A_μ is present on the *transverse* manifold Σ. The only difference is that

$$\slashed{D}_{\Sigma, A_\mu} \Psi = \lambda \Psi, \tag{12}$$

where $\slashed{D}_{\Sigma, A_\mu}$ is the Dirac operator twisted by the the connection A_μ. Assuming that $\psi \propto e^{-i\omega t}$ one obtains the second order equation of the form

$$\frac{d^2\chi}{dy^2} + \omega^2 \chi = A^2(m - \frac{\lambda}{C})^2 \chi. \tag{13}$$

B.   The Background of Spherically Symmetric Static Black Hole

As was shown in Refs. [8, 36] the spectral decomposition method will be fruitful for the analysis of the evolution of massive Dirac hair in the spacetime of static spherically symmetric black hole. The asymptotic tail is connected with the existence of a branch cut situated along the interval $-m \leq \omega \leq m$. An oscillatory inverse power-law behaviour of the massive spinor field arises from the integral of Green function $\tilde{G}(y, y'; \omega)$ around branch cut. The time evolution of the field may be written in the following form:

$$\chi(y, t) = \int dy' \left[G(y, y'; t)\psi_t(y', 0) + G_t(y, y'; t)\psi(y', 0) \right],\tag{14}$$

for $t > 0$, where the Green's function $G(y, y'; t)$ is given by the relation

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + V \right] G(y, y'; t) = \delta(t)\delta(y - y').\tag{15}$$

Next, our main task will be to find the black hole Green function. In the first step we reduce equation (15) to an ordinary differential equation. To do it one can use the Fourier transform [36] $\tilde{G}(y, y'; \omega) = \int_{0^-}^{\infty} dt\, G(y, y'; t)e^{i\omega t}$. This Fourier's transform is well defined for $Im\, \omega \geq 0$, while the corresponding inverse transform yields

$$G(y, y'; t) = \frac{1}{2\pi} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} d\omega\, \tilde{G}(y, y'; \omega)e^{-i\omega t},\tag{16}$$

for some positive number ϵ. The Fourier's component of the Green's function $\tilde{G}(y, y'; \omega)$ can be written in terms of two linearly independent solutions for homogeneous equation as

$$\left(\frac{d^2}{dy^2} + \omega^2 - \tilde{V} \right)\chi_i = 0, \qquad i = 1, 2,\tag{17}$$

where $\tilde{V} = A^2 \left(m - \frac{\lambda}{C} \right)^2$.

The boundary conditions for ψ_i are described by purely ingoing waves crossing the outer horizon H_+ of the n-dimensional static charged black hole $\psi_1 \simeq e^{-i\omega y}$ as $y \to -\infty$ while ψ_2 should be damped exponentially at i_+, namely $\psi_2 \simeq e^{-\sqrt{m^2 - \omega^2}y}$ at $y \to \infty$.

In order to proceed further, we change the variables

$$\chi_i = \frac{\xi}{A^{1/2}\, B^{-1/2}},\tag{18}$$

where $i = 1, 2$.

Although the formalism presented above is true for an arbitrary n-dimensional spacetime we shall consider general form of four-dimensional spherically symmetric static black hole. Black hole under consideration will be characterized by gravitational mass M and some other parameters M', Q, Q' of the background field in addition to the gravitational mass.

We shall consider wave modes only in a far region from the black hole, as a generic behaviours observed in any black hole background. For that purpose, let us assume that $r/M \gg 1$, where M is the gravitational mass of background field. Having all these in mind, the expansion of the metric coefficients $A(r)$ and $B(r)$ as a power series in M/r give us the following:

$$A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \mathcal{O}\left(\frac{1}{r^3} \right),\tag{19}$$

while for the next metric function one has the relation

$$B(r) = 1 + \frac{2M'}{r} + \frac{Q'^2}{r^2} + \mathcal{O}\left(\frac{1}{r^3} \right),\tag{20}$$

where by M' and Q, Q' we have denoted other parameters characterizing the background fields.

Let us assume that the observer and the initial data are situated far away from the considered black hole. We expand Eq. (17) as power series of M/r neglecting terms of order $\mathcal{O}(\omega/r)^2$ and higher. It leads us to the following expression:

$$\frac{d^2}{dr^2}\xi + \left(\omega^2 - m^2\right)\xi \; + \; \left[\frac{2\tilde{\alpha}(\omega^2 - m^2) + \alpha_1}{r}\right. \tag{21}$$

$$\left. - \frac{\omega^2(\tilde{\alpha}^2 + \tilde{\beta}^2) + \beta_1 - m^2\,\tilde{\alpha}^2 - 2\tilde{\alpha}\alpha_1 - 2\tilde{\beta}m^2}{r^2}\right]\xi = 0\,,$$

where

$$\alpha_1 \; = \; 2m(\lambda + Mm), \qquad \beta_1 = \lambda^2 + 4Mm\lambda + Q^2 m^2, \tag{22}$$

$$\tilde{\alpha} \; = \; M + M', \qquad \tilde{\beta} = \frac{Q^2 - Q'^2}{2} + \frac{M'^2 - M^2}{2} + MM'.$$

It turned out that the above equation can be solved in terms of Whittaker's functions, namely the two basic solutions are needed to construct the Green function, with the condition that $\mid \omega \mid \geq m$. Consequently, it implies the result as follows:

$$\tilde{\chi}_1 = M_{\kappa,\tilde{\mu}}(2\tilde{\omega}r), \qquad \tilde{\chi}_2 = W_{\kappa,\tilde{\mu}}(2\tilde{\omega}r)\,, \tag{23}$$

where we have denoted

$$\tilde{\mu} \; = \; \sqrt{1/4 + \omega^2(\tilde{\alpha}^2 + \tilde{\beta}^2) + \beta_1 - m^2\,\tilde{\alpha}^2 - 2\tilde{\alpha}\alpha_1 - 2\tilde{\beta}m^2}, \tag{24}$$

$$\kappa \; = \; \frac{2\tilde{\alpha}(\omega^2 - m^2) + \alpha_1}{2\tilde{\omega}}, \qquad \tilde{\omega}^2 = m^2 - \omega^2.$$

By virtue of the above relations the spectral Green function has the form

$$G_c(x,y;t) \; = \; \frac{1}{2\pi}\int_{-m}^{m} dw \left[\frac{\tilde{\chi}_1(x,\tilde{\omega}e^{\pi i})\,\tilde{\chi}_2(y,\tilde{\omega}e^{\pi i})}{W(\tilde{\omega}e^{\pi i})} - \frac{\tilde{\chi}_1(x,\tilde{\omega})\,\tilde{\chi}_2(y,\tilde{\omega})}{W(\tilde{\omega})}\right] e^{-iwt} \tag{25}$$

$$= \; \frac{1}{2\pi}\int_{-m}^{m} dw f(\tilde{\omega})\, e^{-iwt},$$

where $W(\tilde{\omega})$ is the Wronskian.

First we discuss the intermediate asymptotic behaviour of the massive scalar field, i.e., when the range of parameters are $M \ll r \ll t \ll M/(mM)^2$. The intermediate asymptotic contribution to the Green function integral gives the frequency equal to $\tilde{\omega} = \mathcal{O}(\sqrt{m/t})$. It implies that $\kappa \ll 1$. One should have in mind that κ stems from the $1/r$ term in the massive scalar field equation of motion. Thus it depicts the effect of backscattering off the spacetime curvature and in the case under consideration the backscattering is negligible. In the case of intermediate asymptotic behaviour one finally gets

$$f(\tilde{\omega}) = \frac{2^{2\tilde{\mu}-1}\Gamma(-2\tilde{\mu})\,\Gamma(\frac{1}{2} + \tilde{\mu})}{\tilde{\mu}\Gamma(2\tilde{\mu})\,\Gamma(\frac{1}{2} - \tilde{\mu})}\left[1 + e^{(2\tilde{\mu}+1)\pi i}\right](rr')^{\frac{1}{2}+\tilde{\mu}}\tilde{\omega}^{2\tilde{\mu}}, \tag{26}$$

where we have used the fact that $\tilde{\omega}r \ll 1$ and the form of $f(\tilde{\omega})$ can be approximated by means of the fact that $M(a,b,z) = 1$ as z tends to zero.

However, the resulting Green function can not be calculated analytically because of the fact that the parameter $\tilde{\mu}$ depends on ω. We calculate it numerically and results are provided in Figs 1-5. In Fig. 1 we present the graph of the logarithm of intermediate Green function $\ln \mid G_c(r,r';t) \mid$ versus time t for different mass parameter $M = 0.1, M = 1.0$ and $M = 5.0$. In Fig. 2 we depict the dependence of the function in question on $M' = 0.1, M' = 5.0$, while in Fig. 3 we get the relation among intermediate Green function and Q (for $Q = 0.0$ and $Q = 1.0$). Fig. 4 is connected with the dependence of the intermediate Green function on mass of the Dirac hair. We plotted the graph for $m = 0.005$, $m = 0.01$, $m = 0.02$. We finished with the Fig. 5, where the dependence on the multiple number of the wave mode was revealed for $\lambda = 0.01, \lambda = 0.1$ and $\lambda = 1.0$. Summing it all up one can remark that the intermediate asymptotic Green function depends on the parameters M, M', Q as well as the mass of the Dirac spinor field and the multiple number of the wave mode.

The different pattern of decay is expected when $\kappa \gg 1$, for the late-time behaviour, when the backscattering off the curvature is important. In this case we use the limit

$$M_{\kappa,\tilde{\mu}}(2\tilde{\omega}r) \approx \Gamma(1+2\tilde{\mu})\ (2\tilde{\omega}r)^{\frac{1}{2}}\kappa^{-\tilde{\mu}}\ J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r}). \tag{27}$$

Consequently, $f(\tilde{\omega})$ yields

$$\begin{aligned} f(\tilde{\omega}) \ = \ & \frac{\Gamma(1+2\tilde{\mu})\ \Gamma(1-2\tilde{\mu})}{2\tilde{\mu}}\ (rr')^{\frac{1}{2}}\left[J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r})\ J_{-2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r'}) - I_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r})\ I_{-2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r'})\right] \\ & + \frac{(\Gamma(1+2\tilde{\mu}))^2\ \Gamma(-2\tilde{\mu})\ \Gamma(\frac{1}{2}+\tilde{\mu}-\kappa)}{2\tilde{\mu}\ \Gamma(2\tilde{\mu})\ \Gamma(\frac{1}{2}-\tilde{\mu}-\kappa)}\ (rr')^{\frac{1}{2}}\ \kappa^{-2\tilde{\mu}}\bigg[J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r})\ J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r'}) \\ & + \ e^{(2\tilde{\mu}+1)}I_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r})\ I_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r'})\bigg]. \end{aligned} \tag{28}$$

It can be noticed that the first part of the above Eq. (28) the late time tail is proportional to t^{-1}. Just we calculate the second term of the right-hand side of Eq. (28). For the case when $\kappa \gg 1$ it can be brought to the form written as

$$G_{c\ (2)}(r,r';t) = \frac{M}{2\pi}\int_{-m}^{m}\ dw\ e^{i(2\pi\kappa - wt)}\ e^{i\phi}, \tag{29}$$

where we have defined

$$e^{i\varphi} = \frac{1+(-1)^{2\tilde{\mu}}e^{-2\pi i\kappa}}{1+(-1)^{2\tilde{\mu}}e^{2\pi i\kappa}}, \tag{30}$$

while M provides the relation as follows:

$$M = \frac{(\Gamma(1+2\tilde{\mu}))^2\ \Gamma(-2\tilde{\mu})}{2\tilde{\mu}\ \Gamma(2\tilde{\mu})}\ (rr')^{\frac{1}{2}}\left[J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r})\ J_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r'}) + I_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r})\ I_{2\tilde{\mu}}(\sqrt{8\kappa\tilde{\omega}r'})\right]. \tag{31}$$

At very late time both terms e^{iwt} and $e^{2\pi\kappa}$ are rapidly oscillating. It means that the scalar waves are mixed states consisting of the states with multipole phases backscattered by spacetime curvature. Most of them cancel with each others which have the inverse phase. In such a case, one can find the value of $G_{c(2)}$ by means of the saddle point method. It could be found that the value $2\pi\kappa - wt$ is stationary at the value of w equal to the following:

$$a_0 = \left[\frac{\pi\ (2\tilde{\alpha}(\omega^2 - m^2) + \alpha_1)}{2\sqrt{2}m}\right]^{\frac{1}{3}}, \tag{32}$$

Then approximating integration by the contribution from the very close nearby of a_0 is given by

$$F = \frac{im^{2/3}\sqrt{2}}{3\sqrt{\alpha_1}}(\pi\ \alpha_1)^{\frac{5}{6}}(mt)^{-\frac{5}{6}}\ e^{i\varphi(a_0)}. \tag{33}$$

In comparison to the late-time behaviour of the second term in Eq. (28), the first term can be neglected. The dominant role plays the behaviour of the second term, i.e., the late-time behaviour is proportional to $-\frac{5}{6}$. Thus, the asymptotic late-time behaviour of the Green's function can be written in the form

$$G_c(r,r;t) = \frac{2\sqrt{2}\ m^{2/3}}{\sqrt{3}}\pi^{\frac{5}{6}}\ \alpha_1^{\frac{1}{3}}\ (mt)^{-\frac{5}{6}}\ \sin(mt)\ \tilde{\chi}(r,m)\ \tilde{\chi}(r',m). \tag{34}$$

The above relation provides the main conclusion of our investigations, i.e., analytical proof that the late-time asymptotic decay pattern of massive Dirac field is unaffected by black hole characteristics and has the form of power law pattern proportional to $t^{-5/6}$.

III. CONCLUSIONS

Dirac fields were intensively investigated in various contexts in spacetimes of black holes. Among all, problems concerning quasi-normal modes, high overtones were studied (see e.g., the latest works concerning this widely elaborated

field of research [38]). In our paper we restrict our attention to the problem of the intermediate and late-time decay pattern of massive Dirac hair in the background of static spherically black hole. This problem is strictly bounded with the *no-hair* theorem for the objects in question.

It has been shown in previous papers [13, 14] that the oscillatory power-law decay rate of massive Dirac hair which decay rate is $t^{-5/6}$ dominates at asymptotically late-time in Schwarzschild and RN black hole spacetimes. In our considerations we envisaged the fact that this behaviour was generic one observed in any static spherically symmetric background. We assume that a massive Dirac field propagates in a static spherically symmetric spacetime with asymptotically flat metric. The metric under consideration is described in terms of the ADM mass M and some other parameters M', Q, Q' of the background fields in addition to the afore mentioned gravitational mass. In Refs.[13, 14, 16] the complicated equation of motion for massive Dirac fields were solved by means of the so-called Newman-Penrose formalism. In our research we introduce much easier method of solving Dirac equation. The properties of the Dirac operator enable us to find the second order differential equation which in turn can be applied to construct the spectral Green function, crucial in our investigations. One should remark that our method may be used in arbitrary spacetime dimension contrary to the method of solvig Dirac Eqs. presented in the afore mentioned papers. However, we restrict our research to four-dimensional case.

Unfortunately, the intermediate asymptotic behavior of massive spinor hair can not be calculated analytically. Then, we performed numerical calculations and found the graphs of the logarithm of the intermediate Green function versus time for changes of the parameters like M, M', Q, mass of the Dirac field m and the multiple number of the wave function. We revealed the dependence of the Green function in question on the above parameters.

However, it turned out this is not the final pattern of vanishing the massive spinor hair. The resonance backscattering off the spacetime curvature reveals at very late times. It was proved analytically that the massive spinor decay in the spacetime under consideration is proportional to $t^{-\frac{5}{6}}$. So we conclude that this long-lived oscillating tail is generally observed at timelike infinity in the black hole spacetimes. The same conclusion concerning the pattern of decay was achieved in Ref. [39], where the massive scalar hair decay was elaborated in static spherically symmetric black hole spacetime.

Our work comprises all the previously studied cases of static spherically symmetric black hole spacetimes on which propagates massive Dirac field. Finally, one should stress that we obtain the analytical proof that the late-time decay pattern of massive Dirac spinor hair in the background of general static spherically symmetric four-dimensional black hole it of the form $t^{-\frac{5}{6}}$. It does not depend on gravitational mass and additional parameters characterizing background fields.

It will be interesting to study the decay pattern of higher spin fields in the general static spherically background (the research in this direction was begun in Ref.[17]) and to consider n-dimensional black hole background. We hope to return to these problems elsewhere.

Acknowledgments

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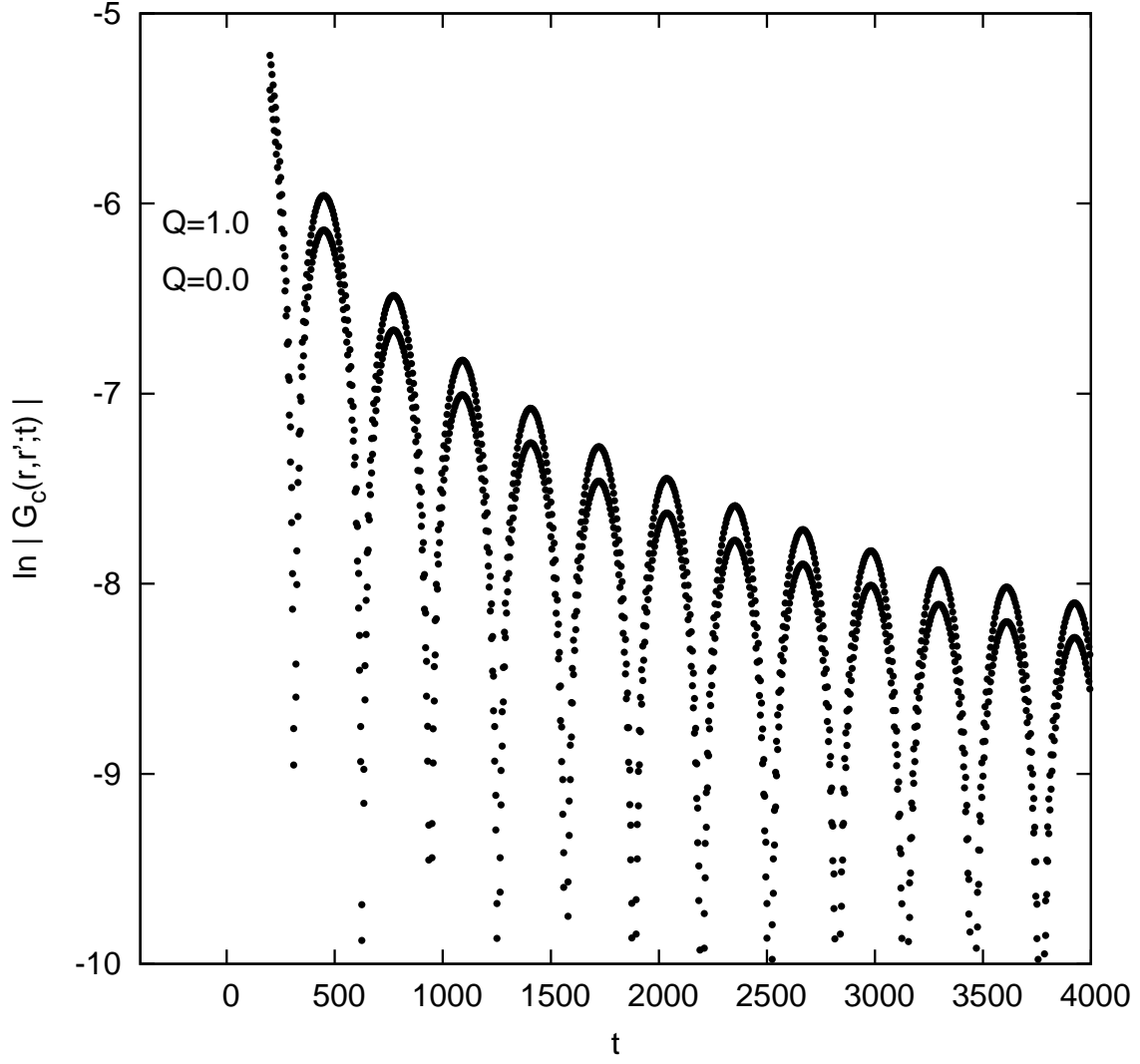[39] H.Koyama and A.Tomimatsu, Phys. Rev. D **65**, 084031 (2002).

FIG. 1: Amplitude of the Green's function vs. time for different Q. Parameters: $M = 1$, $M' = 1$, $Q' = 0$, $rr' = 1000$, $\lambda = 0.01$, $m = 0.01$
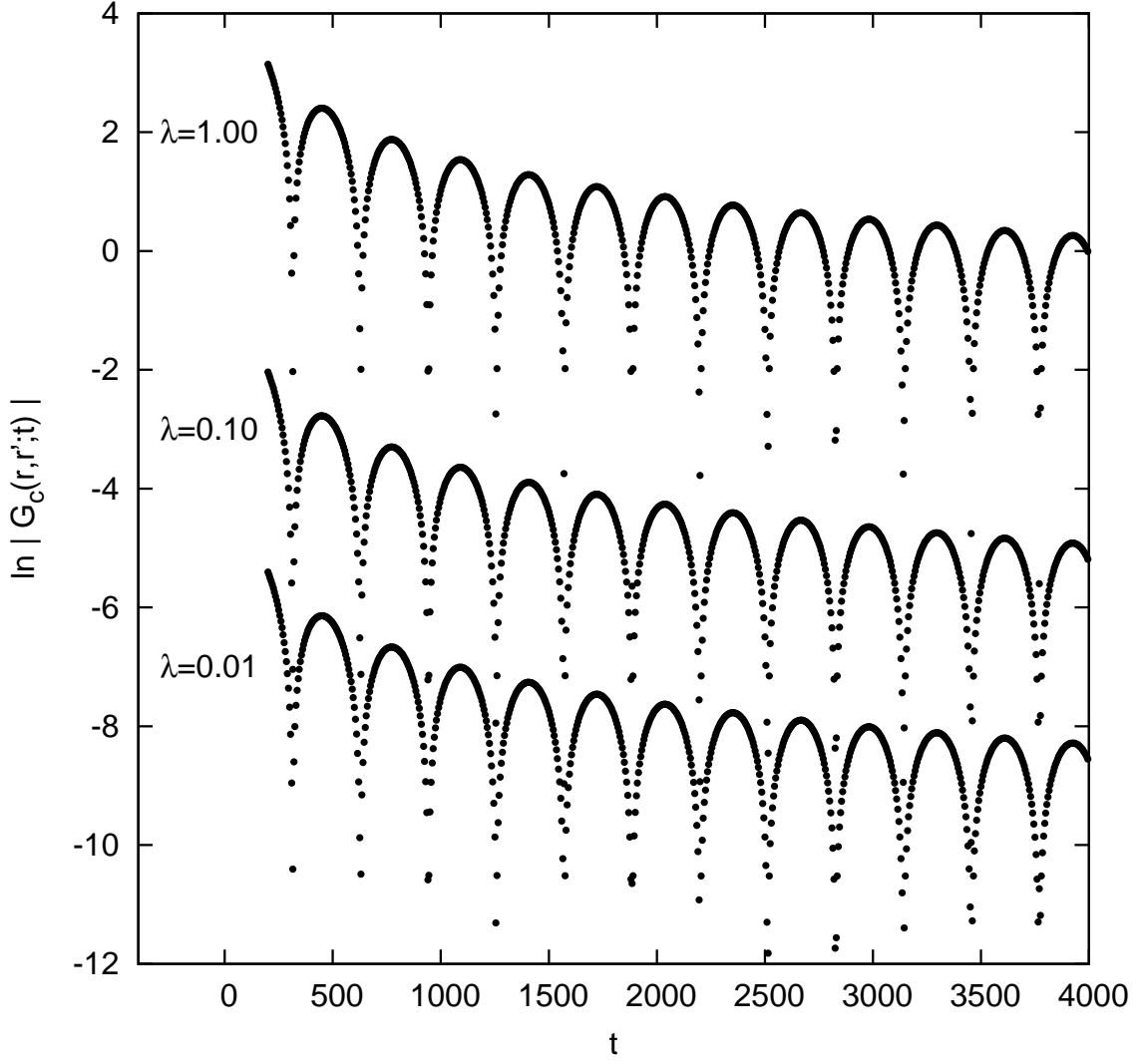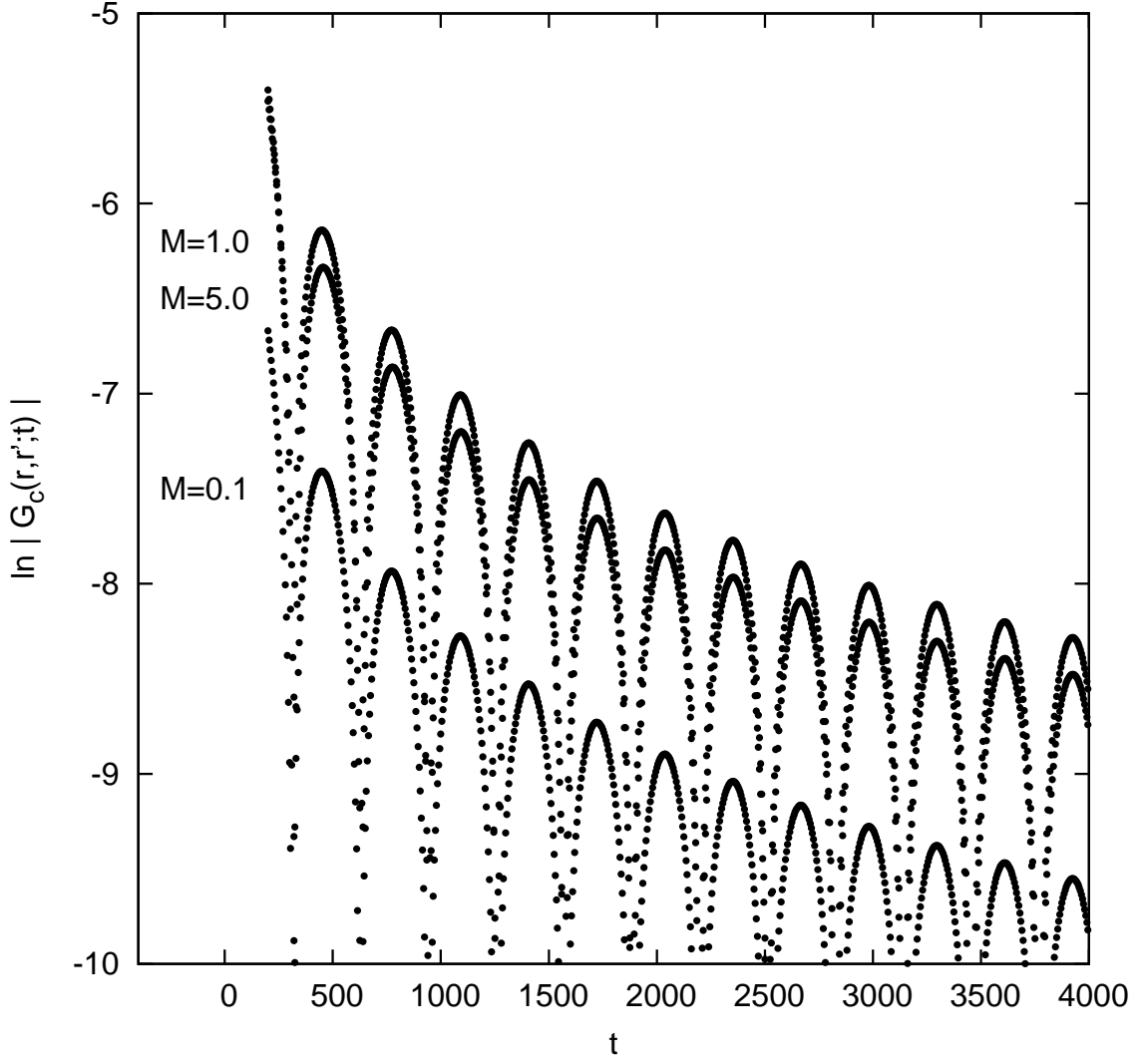
FIG. 2: Amplitude of the Green's function vs. time for different λ. Parameters: $M = 1$, $M' = 1$, $Q = 0$, $Q' = 0$, $rr' = 1000$, $m = 0.01$.

FIG. 3: Amplitude of the Green's function vs. time for different M. Parameters: $M' = 1$, $Q = 0$, $Q' = 0$, $rr' = 1000$, $\lambda = 0.01$, $m = 0.01$.
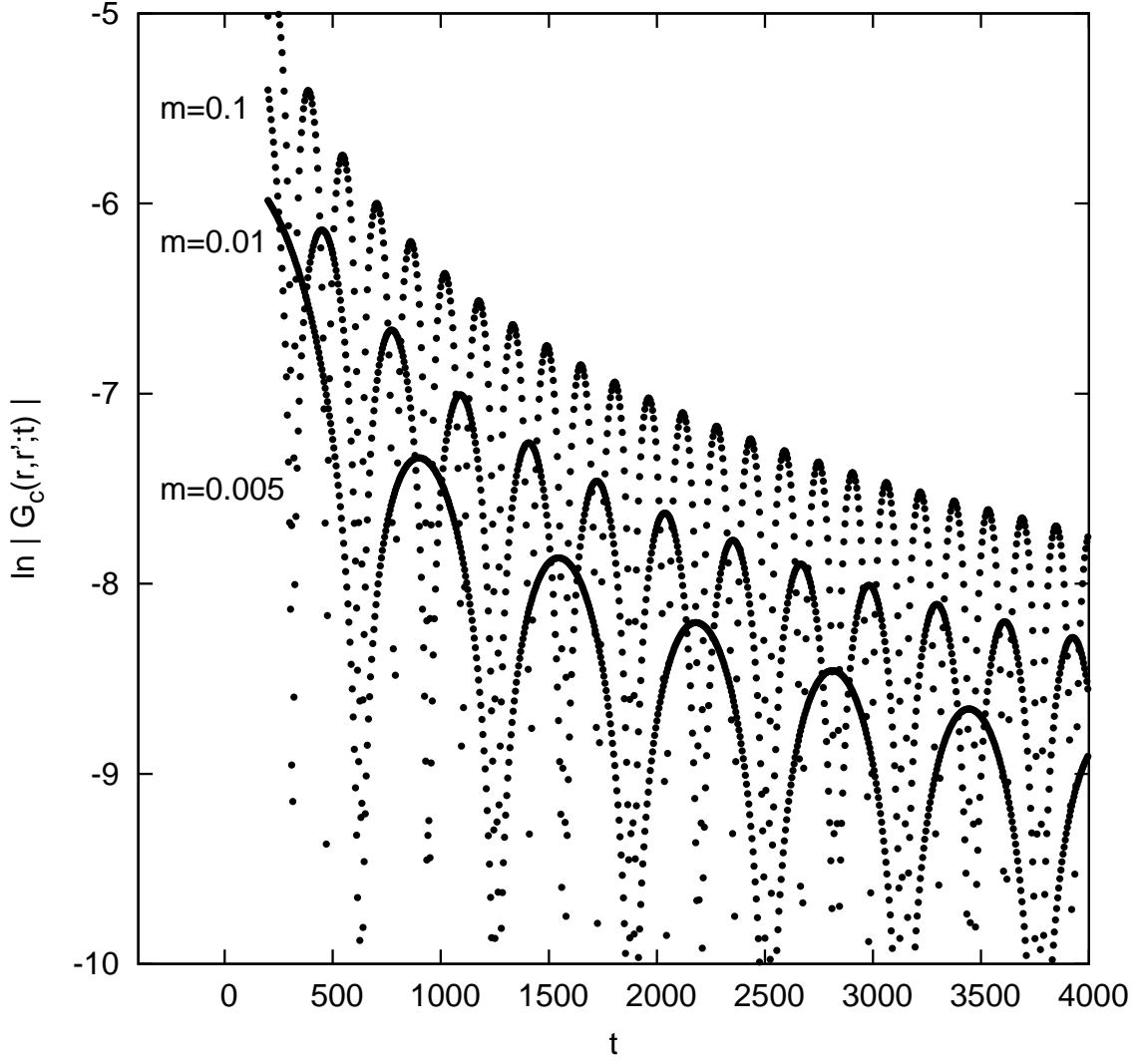
FIG. 4: Amplitude of the Green's function vs. time for different m. Parameters: $M = 1$, $M' = 1$, $Q = 0$, $Q' = 0$, $rr' = 1000$, $\lambda = 0.01$.
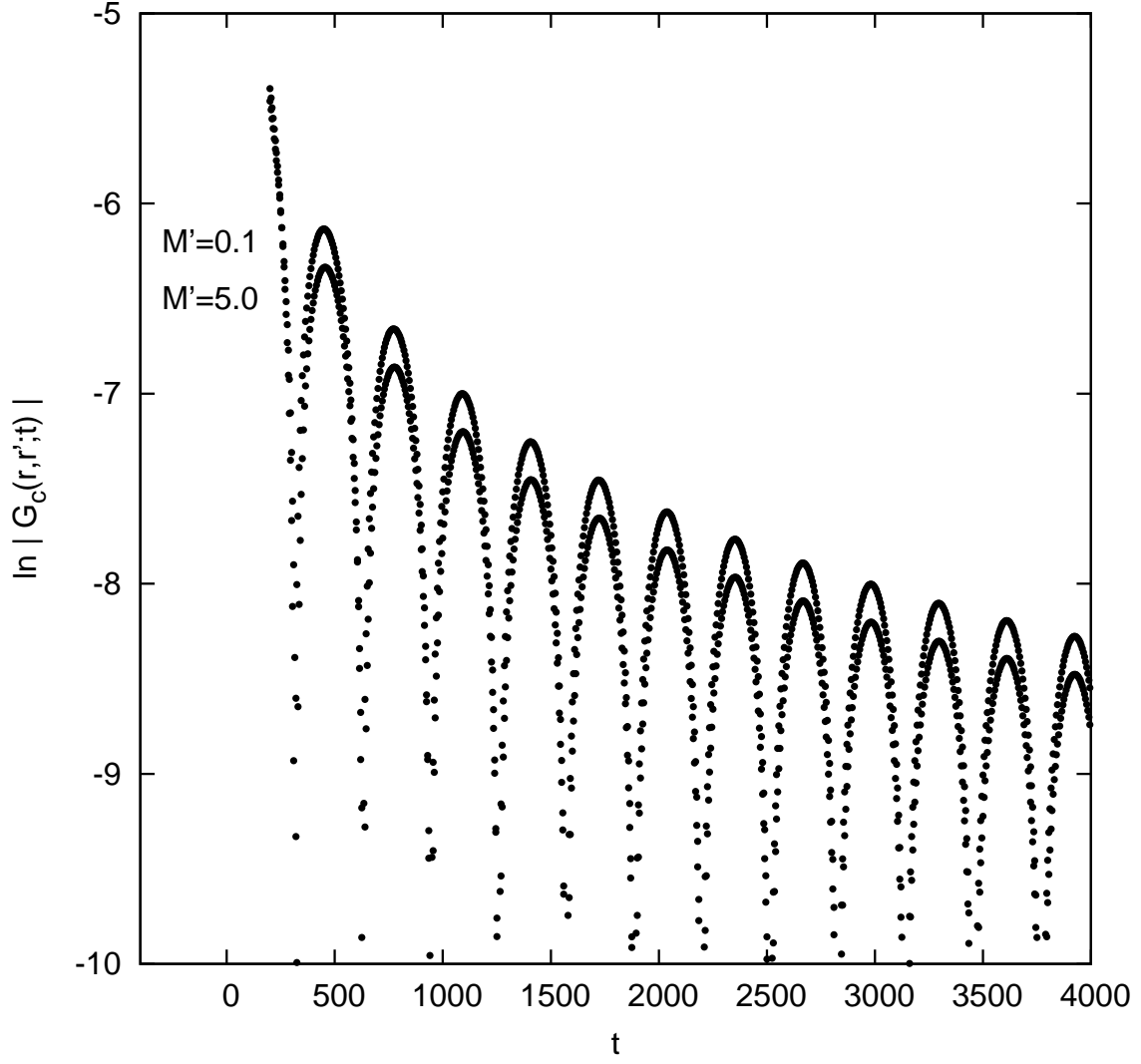
FIG. 5: Amplitude of the Green's function vs. time for different M'. Parameters: $M = 1$, $Q = 0$, $Q' = 0$, $rr' = 1000$, $\lambda = 0.01$, $m = 0.01$.